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$$e_{R_T} = \sqrt{\frac{V_{out}^2}{\left(V_{in} - V_{out}\right)^2} e_{R_1}^2 + \frac{R_1^2 V_{out}^2}{\left(V_{in} - V_{out}\right)^4} e_{V_{in}}^2 + \frac{R_1^2 V_{in}^2}{\left(V_{in} - V_{out}\right)^4} e_{V_{in}}^2}$$

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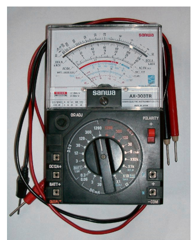

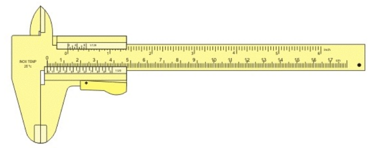

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Lecture 2H – Error Propagation

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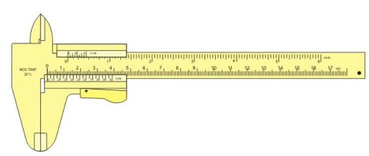
Overview Error Propagation

You've:
1. Made sets of measurements
2. Calculated statistics for them

You now need to:
1. Use the measurements in a formula
2. Estimate uncertainty for the formula result

You can do so:
- Analytically
- Numerically



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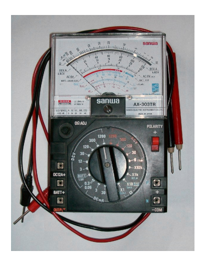

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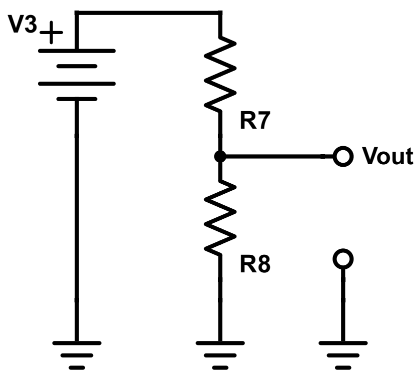
Example Voltage Divider



R7 = 10 kΩ ± 1%

R8 = 20 kΩ ± 1%

V3 = 3.3 V ± 0.05 V

Vout = ? V ± ? V

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Example: Numerical method

Nominal

$$V_{out} = V_3 \frac{R_8}{R_7 + R_8} = 3.3\text{ V}\frac{20\text{ k}\Omega}{(10+20)\text{ k}\Omega} = 2.2\text{ V}$$

R7 High

$$V_7 = 10\text{ k}\Omega + 1\% = (1+0.01)\bullet 10\text{ k}\Omega = 10.1\text{ k}\Omega$$

$$V_{out} = V_3 \frac{R_8}{R_7 + R_8} = 3.3\text{ V}\frac{20\text{ k}\Omega}{(10.1+20)\text{ k}\Omega} = 2.193\text{ V}$$

R7 Low

$$V_7 = 10\text{ k}\Omega - 1\% = (1-0.01)\bullet 10\text{ k}\Omega = 9.9\text{ k}\Omega$$

$$V_{out} = V_3 \frac{R_8}{R_7 + R_8} = 3.3\text{ V}\frac{20\text{ k}\Omega}{(9.9+20)\text{ k}\Omega} = 2.207\text{ V}$$

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Example: Numerical (Cont.)

R8 High & Low

$$R_8 = 20.2\ \text{k}\Omega$$

$$V_{out} = 3.3\ \text{V}\,\frac{20.2\ \text{k}\Omega}{(10+20.2)\ \text{k}\Omega} = 2.207\ \text{V}$$

$$R_8 = 19.8\ \text{k}\Omega$$

$$V_{out} = 3.3\ \text{V}\,\frac{19.8\ \text{k}\Omega}{(10+19.8)\ \text{k}\Omega} = 2.193\ \text{V}$$

V3 High & Low

$$V_3 = 3.25\ \text{V}$$

$$V_{out} = 3.25\ \text{V}\,\frac{20\ \text{k}\Omega}{(10+20)\ \text{k}\Omega} = 2.167\ \text{V}$$

$$V_3 = 3.35\ \text{V}$$

$$V_{out} = 3.35\ \text{V}\,\frac{20\ \text{k}\Omega}{(10+20)\ \text{k}\Omega} = 2.233\ \text{V}$$

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Example: Numerical (Cont.)

For R7 high $V_{out} - V_{out-nom} = 2.193\ \text{V} - 2.200\ \text{V} = -0.007\ \text{V}$

For R8 high $V_{out} - V_{out-nom} = 0.007\ \text{V}$

For V3 high $V_{out} - V_{out-nom} = 0.033\ \text{V}$

For R7 low $V_{out} - V_{out-nom} = 0.007\ \text{V}$

For R8 low $V_{out} - V_{out-nom} = -0.007\ \text{V}$

For V3 low $V_{out} - V_{out-nom} = -0.033\ \text{V}$

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E80 Experimental Engineering — Example: Numerical (Cont.)

Root-Sum-of-Squares Addition

$$+\varepsilon_{Vout} = +\sqrt{(-0.007)^2 + 0.007^2 + 0.033^2} = 0.035$$

$$V_{out} = 2.200 \pm 0.035 \text{ V}$$

If we knew the confidence limits, we would report

$$V_{out} = 2.200 \pm 0.035 \text{ V(95\% conf)}$$

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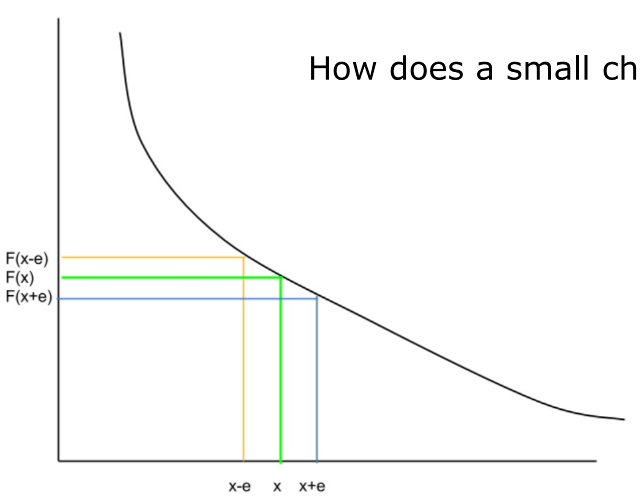
E80 Experimental Engineering — Function Perturbation

How does a small change in x affect $F(x)$?



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Analytical Derivation

For $F = F(x, y, x, \cdots)$ e.g., $V_{out} = V_{out}(R_7, R_8, V_3) = V_3 \dfrac{R_8}{R_7 + R_8}$

Taylor series expansion

$$F - F_{true} = \frac{\partial F}{\partial x}(x - x_{true}) + \frac{\partial F}{\partial y}(y - y_{true}) + \frac{\partial F}{\partial z}(z - z_{true}) + \cdots$$

Let $\varepsilon_x = x - x_{true}, \cdots$ Then $\varepsilon_F = \dfrac{\partial F}{\partial x}\varepsilon_x + \dfrac{\partial F}{\partial y}\varepsilon_y + \dfrac{\partial F}{\partial z}\varepsilon_z + \cdots$

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In Our Case

$$V_{out} = V_{out}(R_7, R_8, V_3) = V_3 \frac{R_8}{R_7 + R_8}$$ — The Function

$$\frac{\partial V_{out}}{\partial R_7} = -\frac{V_3 R_8}{\left(R_7 + R_8\right)^2} = -\frac{1}{R_7 + R_8} V_{out}$$

$$\frac{\partial V_{out}}{\partial R_8} = \frac{V_3 R_7}{\left(R_7 + R_8\right)^2} = \frac{R_7}{R_8}\frac{1}{R_7 + R_8} V_{out}$$

$$\frac{\partial V_{out}}{\partial V_3} = \frac{R_8}{R_7 + R_8} = \frac{1}{V_3} V_{out}$$

} The Partial Derivatives

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Example: Analytical

In the general case $$\varepsilon_F = \frac{\partial F}{\partial x}\varepsilon_x + \frac{\partial F}{\partial y}\varepsilon_y + \frac{\partial F}{\partial z}\varepsilon_z + \cdots$$

In our case $$\varepsilon_{Vout} = \frac{\partial V_{out}}{\partial R_7}\varepsilon_{R7} + \frac{\partial V_{out}}{\partial R_8}\varepsilon_{R8} + \frac{\partial V_{out}}{\partial V_3}\varepsilon_{V3}$$

$$= \left[-\frac{V_3 R_8}{\left(R_7 + R_8\right)^2}\right]\varepsilon_{R7} + \frac{V_3 R_7}{\left(R_7 + R_8\right)^2}\varepsilon_{R8} + \frac{R_8}{R_7 + R_8}\varepsilon_{V3}$$

$$= \left[\left(-\frac{1}{R_7 + R_8}\right)\varepsilon_{R7} + \frac{R_7}{R_8}\frac{1}{R_7 + R_8}\varepsilon_{R8} + \frac{1}{V_3}\varepsilon_{V3}\right]V_{out}$$

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Root Sum of Squares

In the general case $$\varepsilon_F = \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 \varepsilon_x^2 + \left(\frac{\partial F}{\partial y}\right)^2 \varepsilon_y^2 + \left(\frac{\partial F}{\partial z}\right)^2 \varepsilon_z^2 + \cdots}$$

In our case $$\varepsilon_{Vout} = \sqrt{\left(\frac{\partial V_{out}}{\partial R_7}\right)^2 \varepsilon_{R7}^2 + \left(\frac{\partial V_{out}}{\partial R_8}\right)^2 \varepsilon_{R8}^2 + \left(\frac{\partial V_{out}}{\partial V_3}\right)^2 \varepsilon_{V3}^2}$$

$$= V_{out}\sqrt{\left(-\frac{1}{R_7 + R_8}\right)^2 \varepsilon_{R7}^2 + \left(\frac{R_7}{R_8}\frac{1}{R_7 + R_8}\right)^2 \varepsilon_{R8}^2 + \left(\frac{1}{V_3}\right)^2 \varepsilon_{V3}^2}$$

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E80 Experimental Engineering — Plugging In

$$\varepsilon_{Vout} = 2.2\sqrt{\left(-\frac{1}{10\text{k}+20\text{k}}\right)^2 0.1\text{k}^2 + \left(\frac{10\text{k}}{20\text{k}}\frac{1}{10\text{k}+20\text{k}}\right)^2 0.2\text{k}^2 + \left(\frac{1}{3.3}\right)^2 0.05^2}$$

$$= 2.2\text{ V}\sqrt{\left(\frac{1}{300}\right)^2 + \left(\frac{1}{2}\frac{2}{300}\right)^2 + \left(\frac{5}{330}\right)^2} = 0.0349\text{ V}$$

$$V_{out} = 2.200 \pm 0.035\text{ V}$$

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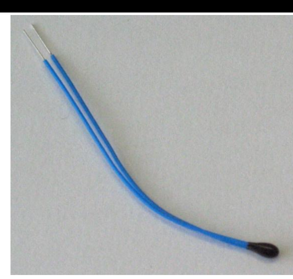
E80 Experimental Engineering — Example: A Thermistor

Governing equation $T = \dfrac{1}{\dfrac{1}{T_0} + \dfrac{1}{\beta}\ln\dfrac{R}{R_0}}$



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Analytical (differential) form

$$dT = \frac{\left[\left(\frac{1}{\beta^2}\ln\frac{R}{R_0}\right)d\beta - \frac{1}{\beta R}dR\right]}{\left[\frac{1}{T_0} + \frac{1}{\beta}\ln\frac{R}{R_0}\right]^2}$$

$$dT = T^2\left[\left(\frac{1}{\beta^2}\ln\frac{R}{R_0}\right)d\beta - \frac{1}{\beta R}dR\right]$$

Subbing in for T

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Errors using Analytical Method

$$e_T = T^2 \left[\left(\frac{1}{\beta^2} \ln \frac{R}{R_0} \right)^2 e_\beta^2 + \left(\frac{1}{\beta R} \right)^2 e_R^2 \right]^{1/2}$$

$$T = 273.14 \pm 1.77 \text{ K}$$

	Nom. Value	error%	error	error term
β	4261	1%	42.61	0.23
R (Ω)	3,700,000	10%	370000	1.75
T (K)	273.14			**1.77**
T_0 (K)	298.15			
R_0 (Ω)	1000000			

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Errors using Numerical Method

$$T = \frac{1}{\frac{1}{T_0} + \frac{1}{\beta} \ln \frac{R}{R_0}}$$

Value	Nominal	β + 1%	β − 1%	R + 10%	R − 10%
T_0 (K)	298.15	298.15	298.15	298.15	298.15
R_0 (Ω)	1,000,000	1,000,000	1,000,000	1,000,000	1,000,000
β	4261	4303.61	4218.39	4261	4261
R (Ω)	3,700,000	3,700,000	3,700,000	4,070,000	3,330,000
T (K)	273.14	273.37	272.91	271.49	275.00
ΔT (K)		0.46		-3.52	
error (K)	**3.55**				
±error (K)	**1.77**				

$$T = 273.14 \pm 1.77 \text{ K}$$

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Trade Offs

- Analytical method
  - Requires partial derivatives
  - Provides insight to relative contributions
  - Much simpler calculations (spreadsheet)
- Numerical method
  - No calculus
  - Less insight into contributions
  - More unwieldly calculations (spreadsheet)

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Relative Magnitudes

Can we neglect error terms?
When?

$$\varepsilon_F = \sqrt{\varepsilon_x^2 + (0.1\varepsilon_x)^2} = \sqrt{1.01\varepsilon_x^2} = 1.005|\varepsilon_x| \approx |\varepsilon_x|$$

Any individual error contribution (the uncertainty times the partial derivative) can be neglected if its absolute value is 10% or less of the largest contribution.

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E80 Experimental Engineering — Rules of Thumb

- Use nominal values or calculated means in formulas.
- Choose the method you understand best.
- Neglect any error terms that are smaller than 10% of the maximum error term.

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E80 Experimental Engineering — Takeaways

1. If you are reporting the results of a calculation that involves inputs with uncertainties, you need to propagate errors and report the uncertainty in the result.
2. You can calculate the uncertainty numerically just from the formula and lots of calculations.
3. You can calculate the uncertainty analytically using partial derivatives and many fewer calculations.
4. Neglect any error terms smaller than 10% of the maximum.

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